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ROYAL AIRCRAFT ESTABLISHMENT, FARNBOROUGH

The descent of an earth-satellite through the upper atmosphere

by

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SUMMARY

A theoretical study is made of the flight path of an uncontrolled but aerodynamically stable satellite as it spirals down through the earth's atmosphere from an initially circular orbit under the action of air drag. The earth and its atmosphere are taken as spherically symmetrical. For altitudes above about 125 n. miles a simple solution is found: the velocity of the satellite is independent of its size, shape, weight and initial altitude and equal to the orbital velocity appropriate to its current altitude; while its angle of descent, in radians is twice its drag/weight ratio. Estimates of lifetime are also made.

1 Introduction

Now that earth-satellites are emerging from the domain of fancy into the colder climate of reality, the citizens who live beneath their orbits will be asking sharper questions about the satellites' probable behaviour when, under the prolonged action of air drag, they come down to earth. A 20 lb satellite seems trivial enough, but the average householder might be somewhat daunted at the prospect of a 20 ton satellite, red-hot, landing in his garden. The likelihood of such a landing depends on the speed and angle of descent of the satellite at the altitudes where severe aerodynamic heating occurs; and this in turn depends on the past history of the satellite, on how it behaves as it spirals down through the upper atmosphere. In this Memo. it is shown that at altitudes above about 125 n. miles simple and general formulae for the velocity and angle of descent can be derived, if certain simplifying assumptions, stated below, are accepted.

2 Assumptions

The two main assumptions are:

(i) The earth is spherical and non-rotating, with a spherically symmetrical atmosphere and gravitational field; and the satellite is initially in a circular orbit.

(ii) The satellite is uncontrolled and perfectly stable aerodynamically, remaining at zero incidence throughout its descent path. Thus its axis points in the direction of motion and the only force acting, other than gravity, is axial drag.

These assumptions are helpful simplifications in a first attack on the problem and it is difficult to guess how serious are the errors incurred. In practice the earth's oblateness, or rather the atmosphere's oblateness, would impose a periodic variation on the axial drag. Also, unless the satellite was a sphere its incidence would not be zero but would vary periodically in a weathercock oscillation of low frequency; and superposed on this oscillation would be another resulting from the rotation of the atmosphere. Intuitively one would expect that these various disturbances would nearly impose small perturbations on the simplified solution derived here, though there is always the risk that the perturbations may build up through resonance.

3 Analysis

Fig.1 shows the notation. CO is a fixed line through the initial position of the satellite and S is its position at any later time t. The two forces acting on the satellite are drag D parallel to its axis, and weight mgR^2/r^2, where m is its mass, g the acceleration due to gravity at the earth's surface, R the earth's radius and r the distance CS between satellite and earth's centre. Let ϕ be the angle between CS and CO, let v be the satellite's velocity, and let θ be its angle of descent, i.e. the inclination of its flight path to the local horizontal.

Then, resolving tangential and normal to the flight path, we have, as the dynamical equations of motion,

$$m\frac{dv}{dt} = \frac{mgR^2}{r^2}\sin\theta - D \tag{1}$$

$$mv\frac{d}{dt}(\phi+\theta) = \frac{mgR^2}{r^2}\cos\theta , \tag{2}$$

since $(\phi + \theta)$ is the angle between the direction of motion and the fixed line OB. Equations (1) and (2) are supplemented by two kinematic equations expressing the radial and transverse velocities, namely

$$\frac{dr}{dt} = -v\sin\theta \tag{3}$$

$$r\frac{d\phi}{dt} = v\cos\theta \tag{4}$$

If we assume that θ is small, so that $\sin\theta$ can be taken as θ and $\cos\theta$ as 1, and use equation (4) to eliminate t, equations (1) - (3) become

$$v\frac{dv}{d\phi} = \frac{gR^2\theta}{r} - \frac{rD}{m} \tag{5}$$

$$v^2\frac{d\theta}{d\phi} = \frac{gR^2}{r} - v^2 \tag{6}$$

$$\frac{dr}{d\phi} = -r\theta \tag{7}$$

Dividing equations (5) and (6) by equation (7) to eliminate ϕ:

$$v\frac{dv}{dr} = -\frac{gR^2}{r^2} + \frac{D}{m\theta} \tag{8}$$

$$v^2\theta\frac{d\theta}{dr} = -\frac{gR^2}{r^2} + \frac{v^2}{r} \tag{9}$$

As long as θ is small, the velocity of the satellite is likely to be near the orbital speed appropriate to its altitude, $\sqrt{gR^2/r}$: for if v were very much less than orbital speed the satellite would promptly 'fall out of its orbit', i.e. θ would no longer be small. So it is logical to make the substitution

$$v^2 = \frac{gR^2}{r} - \lambda \tag{10}$$

and to assume that

$$\lambda << \frac{gR^2}{r} \tag{11}$$

Eliminating v from equation (9) by means of equation (10)

$$\lambda = -r\theta\frac{d\theta}{dr}\left(\frac{gR^2}{r} - \lambda\right)$$

$$= -\frac{gR^2}{2}\frac{d}{dr}(\theta^2) \quad \text{to the first order} \tag{12}$$

since $\lambda << gR^2/r$. Differentiating equation (10), we have

$$2v\frac{dv}{dr} = -\frac{gR^2}{r^2} - \frac{d\lambda}{dr} \tag{13}$$

Eliminating $v\,dv/dr$ from (8) and (13):

$$\frac{gR^2}{r^2} - \frac{2D}{m\theta} = \frac{d\lambda}{dr}$$

$$= -\frac{gR^2}{2}\frac{d^2}{dr^2}(\theta^2) \tag{14}$$

by equation (12). To solve equation (14) we assume first that since θ is small the right-hand side of (14) is almost negligible, so that

$$\theta \simeq \frac{2Dr^2}{mgR^2} = 2\,\frac{D}{mgR^2/r^2} = 2\,\frac{\text{drag}}{\text{weight}} \tag{15}$$

Adopting the usual aerodynamic notation we now define the drag coefficient C_D by

$$C_D = \frac{D}{\frac{1}{2}\rho v^2 S} \tag{16}$$

where ρ is the density of the air around the satellite, and S is a reference area, taken as the maximum cross-sectional area of the satellite perpendicular to the direction of motion.

With this notation equation (15) becomes

$$\theta \simeq \rho r\left(\frac{SC_D}{m}\right)\left(\frac{v^2r}{gR^2}\right) = \Delta\left(\frac{v^2r}{gR^2}\right)\rho r \tag{17}$$

if we denote SC_D/m by Δ. Since C_D is constant at high Mach numbers in free-molecule flow, Δ may be taken as constant. Also v^2r/gR^2 is nearly constant by equations (10) and (11), while ρr varies enormously with altitude. So, using (17), we may write

$$\frac{d^2}{dr^2}(\theta^2) \simeq \Delta^2\left(\frac{v^2r}{gR^2}\right)^2\frac{d^2}{dr^2}(\rho r)^2 = \Delta^2\left(\frac{v^2r}{gR^2}\right)^2 f'', \tag{18}$$

where $f = (\rho r)^2$ and dashes denote differentiation with respect to r.

Equation (14) may be re-written

$$\frac{2D}{m\theta} = \frac{gR^2}{r^2}\left\{1 + \frac{r^2}{2}\frac{d^2}{dr^2}(\theta^2)\right\}$$

and, on using (16) and (18) this becomes

$$\frac{\rho v^2}{\theta}\cdot\frac{SC_D}{m} = \frac{gR^2}{r^2}\left\{1 + \frac{r^2\Delta^2}{2}\left(\frac{v^2r}{gR^2}\right)^2 f''\right\}$$

or, since $(v^2r/gR^2) \simeq 1$,

$$\theta = \rho r\,\Delta\,\frac{v^2r}{gR^2}\left(1 - \frac{r^2f''\Delta^2}{2}\right) \tag{19}$$

to the first order.

Also in equation (12) we may take

$$\frac{d}{dr}(\theta^2) = \Delta^2 \left(\frac{v^2 r}{gR^2}\right)^2 f' ,$$

so that, from (10) and (12)

$$v^2 = \frac{gR^2}{r}\left\{1 + \frac{rf'\Delta^2}{2}\right\} \qquad (20)$$

to the first order, on taking $v^2r/gR^2 = 1$ in the second (small) term.

Equations (19) and (20) are the first order solutions for θ and v.

In order to evaluate f' and f'' we must specify a "standard atmosphere", i.e. fix the variation of ρ with r. At present however the air density at high altitudes is not known at all accurately, and any chosen "standard atmosphere" will no doubt prove erroneous when the true values are known. The values chosen here for ρ between 100 and 200 n. miles altitude are given in the table below. With these values of ρ, the quantity ρr can be approximated by the function

$$\rho r \simeq e^{2674.160 - 236.2804r \times 10^{-6} + 5.195r^2 \times 10^{-12}} \qquad (21)$$

the maximum error, for altitudes between 100 and 200 n. miles, being about 3%, as the Table shows. Values of rf' and r^2f'', found by differentiating $f = (\rho r)^2$ as given by (21), are also included in the Table.

TABLE - The variation with altitude of ρ, ρr, etc.

Altitude n. miles	100	120	140	160	180	200
ρ lb/cu ft	471×10^{-12}	103×10^{-12}	27.9×10^{-12}	8.65×10^{-12}	3.18×10^{-12}	1.30×10^{-12}
ρr lb/sq ft	10.1×10^{-3}	2.23×10^{-3}	607×10^{-6}	189×10^{-6}	69.9×10^{-6}	28.8×10^{-6}
Approx. for ρr (21)	10.1×10^{-3}	2.29×10^{-3}	605×10^{-6}	188×10^{-6}	67.7×10^{-6}	28.7×10^{-6}
$-rf'$	55.6×10^{-3}	2.61×10^{-3}	164×10^{-6}	13.9×10^{-6}	1.56×10^{-6}	0.235×10^{-6}
r^2f''	31.6	1.35	76.7×10^{-3}	5.83×10^{-3}	578×10^{-6}	75.8×10^{-6}

The Table provides values of rf' and r^2f'' for substitution in equations (19) and (20): it only remains to choose values for Δ. At altitudes above about 100 n. miles the mean free path of the air molecules is greater than the dimensions of the satellite and 'free-molecule' aerodynamics must be used. In these conditions the drag coefficient C_D of the satellite, whatever its shape, may conveniently be taken as 2 as long as the satellite's speed is much greater than the mean molecular speed. So $\Delta = SC_D/m = 2S/m$, where S is the satellite cross-sectional area in sq feet and m is its mass in lb. Next we must set limits to S/m.

If we exclude objects like metal-foil 'balloons', and also exclude anything smaller than the U.S. 20" sphere of mass 20 lb, we may take 0.1 as an upper limit for S/m. As an example at the other end of the scale we may take a 10,000 lb satellite with a cross-sectional area of 50 sq ft ($\simeq$ 8 ft diameter), giving $S/m \simeq 0.005$. Thus we assume here that Δ lies between 0.01 and 0.2.

If the simplest solutions for v and θ, namely

$$v^2 = \frac{gR^2}{r} \tag{22}$$

and

$$\theta = \rho r\Delta , \tag{23}$$

are to be reasonably accurate, the terms $r^2f''\Delta^2/2$ and $rf'\Delta^2/2$ in equations (19) and (20) must be small. The Table shows that $|rf'|$ is always less than $^1/300$ of r^2f'', and so the error in equation (22) will always be much less than the error in equation (23). If equation (23) is to hold with say 1% accuracy, $r^2f''\Delta^2$ must be less than 0.02, and using the values of r^2f'' in the Table, we find that the maximum permissible values for Δ are 0.025 at 100 n. miles altitude, 0.12 at 120 n. miles altitude and 0.51 at 140 n. miles. Thus the values of θ given by equation (23) are in error by less than 1% down to about 127 n. miles altitude if $\Delta = 0.2$ or about 90 n. miles if $\Delta = 0.01$. Under the same conditions the values of v given by (22) are in error by less than 0.005%. Equation (23) can be re-written in the same form as (15),

$$\theta = 2\frac{\text{drag}}{\text{weight}} ,$$

for

$$\rho r\Delta = \frac{\rho r S C_D}{m} = 2\frac{\frac{1}{2}\rho v^2 S C_D}{mgR^2/r^2} ,$$

since

$$v^2 = gR^2/r .$$

4 Results

At altitudes where equations (22) and (23) hold, i.e. above about 90 n. miles altitude if Δ is small ($\simeq$ 0.01) or above about 125 n. miles if Δ is large ($\simeq$ 0.2), the speed of the satellite at any altitude is equal to the orbital speed appropriate to that altitude, $\sqrt{gR^2/r}$, and the angle of descent θ (the inclination of the flight path to the local horizontal) is very nearly equal to $\rho r\Delta$, or twice the drag/weight ratio, the maximum error being 1%.

Fig.2 shows how the speed varies with altitude, and Fig.3 gives the variation of θ with altitude between 200 and 100 n. miles, accepting the values of ρ given in the Table. At 130 n. miles altitude the speed has reached 25,460 ft/sec, while the angle of descent lies between 0.00066° for $\Delta = 0.01$ and 0.0132° for $\Delta = 0.2$.

The contrast between Fig.2 and Fig.3 is interesting. The satellite's speed is independent of its shape, size and weight, i.e. is independent of Δ, whereas its angle of descent is directly proportional to Δ and hence

inversely proportional to the mass/area ratio m/S. The increase in air density as the satellite descends causes a rapid increase in the angle of descent, which is proportional to the density. But the increase in drag as the satellite descends does not reduce the speed, as might be expected. On the contrary, the speed goes on rising steadily, because the angle of descent always adjusts itself so that half the potential energy accruing from loss in altitude is absorbed by the drag while the other half goes to increase the kinetic energy. This increase in speed as the satellite descends, which of course implies a shorter period of revolution, does not go on indefinitely. The velocity reaches a maximum (at about 80 n. miles altitude when $\Delta = 0.02$) and then falls rapidly.

Once the angle of descent θ is known, the number of revolutions N made by the satellite in dropping from any given initial altitude can easily be found by integration of equation (7) in the form

$$dN = \frac{d\phi}{2\pi} = -\frac{dr}{2\pi r\theta}$$

which shows incidentally that N is proportional to $^1/\theta$, i.e. to $^1/\Delta$, for given r. The number of revolutions performed by satellites having various values of Δ and an initial altitude of 200 n. miles are shown in Fig.4, a scale of days also being included. From this graph the number of revolutions completed between any two altitudes (in the 100-200 n. miles bracket) can be found: e.g. for $\Delta = 0.04$, $N \simeq 750$ revolutions at 150 n. miles, and $N \simeq 810$ at 120 n. miles, i.e. about 60 revolutions are made in descending from 150 to 120 n. miles altitude. It is clear from Fig.4 that very few more revolutions will be performed below 100 n. miles; so the time taken to drop to 100 n. miles can conveniently be taken as the lifetime of the satellite. Lifetime, so defined, is plotted against initial altitude for various Δ in Fig.5. The lifetimes plotted in Fig.5 are in fair agreement with those given in reference 1, which were derived in a different way.

The values of v and θ given in Figs.2 and 3 are those obtained from the simplest solutions, equations (22) and (23). v and θ can be found with similar accuracy down to an altitude some 7 n. miles lower by using equations (20) and (19) instead of (22) and (23). The lower limit of altitude can be pushed a mile or two further still if the second-order approximations for v and θ are used. These are found to be

$$v = \sqrt{\frac{gR^2}{r}}\left\{1 + \frac{rf'\Delta^2}{4} - \frac{r^3\Delta^4}{4}\left(f'f'' + ff'''\right)\right\}$$

$$\theta = \rho r\Delta\left\{1 - \frac{r^2f''\Delta^2}{2} + \frac{r^4\Delta^4}{2}\left(\frac{3}{2}f'' + 2f'f''' + ff''''\right)\right\}$$

if certain terms which can be shown to be small are ignored.

REFERENCE

No.	Author	Title, etc.
1	N.V. Petersen	Lifetimes of satellites in near-circular and elliptic orbits. Jet Propulsion, May 1956, p.341.

Attached

List of symbols
Drgs. Nos. GW/P/7489-7493

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CGWL	
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Director	
DD(E)	
DD(A)	
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LIST OF SYMBOLS

Symbol			Unit
C_D	=	drag coefficient $= D/\frac{1}{2}\rho v^2 S$, taken as 2.	
D	=	drag	pdls
e	=	2.71828....	
f	=	$(\rho r)^2$	lb^2/ft^4
g	=	acceleration due to gravity at earth's surface, taken as 32.2	ft/sec^2
m	=	mass of satellite	lb
N	=	number of revolutions	
r	=	distance of satellite from earth's centre	ft
R	=	earth's radius (taken as $10,800/\pi$ n. miles, or 20.9015×10^6 ft)	
S	=	cross-sectional area of satellite, in plane perpendicular to its axis	sq ft
t	=	time	sec
v	=	velocity of satellite	ft/sec
Δ	=	SC_D/m	ft^2/lb
θ	=	angle of descent of satellite = inclination of flight path to local horizontal	
λ	=	$gR^2/r - v^2$	ft^2/sec^2
π	=	3.14159....	
ρ	=	air density	lb/ft^3
ϕ	=	angular travel of satellite relative to earth's centre (see Fig.1)	

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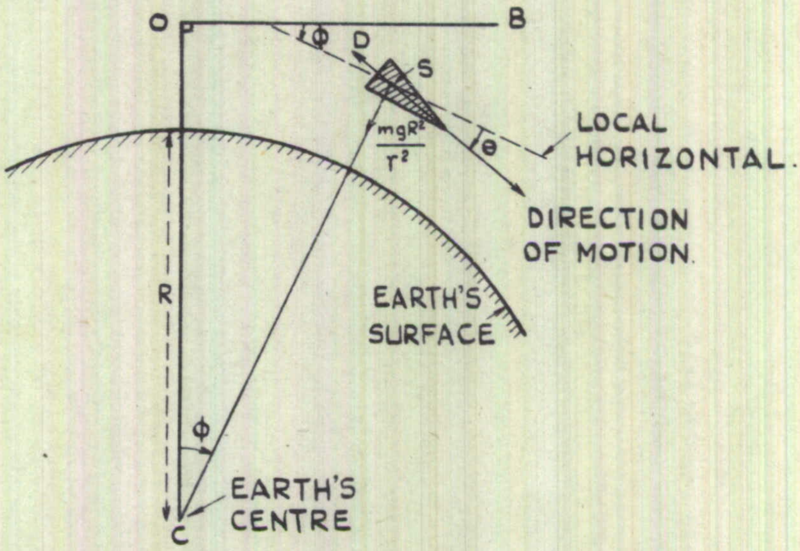

FIG.1. NOTATION.

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FIG. 2.

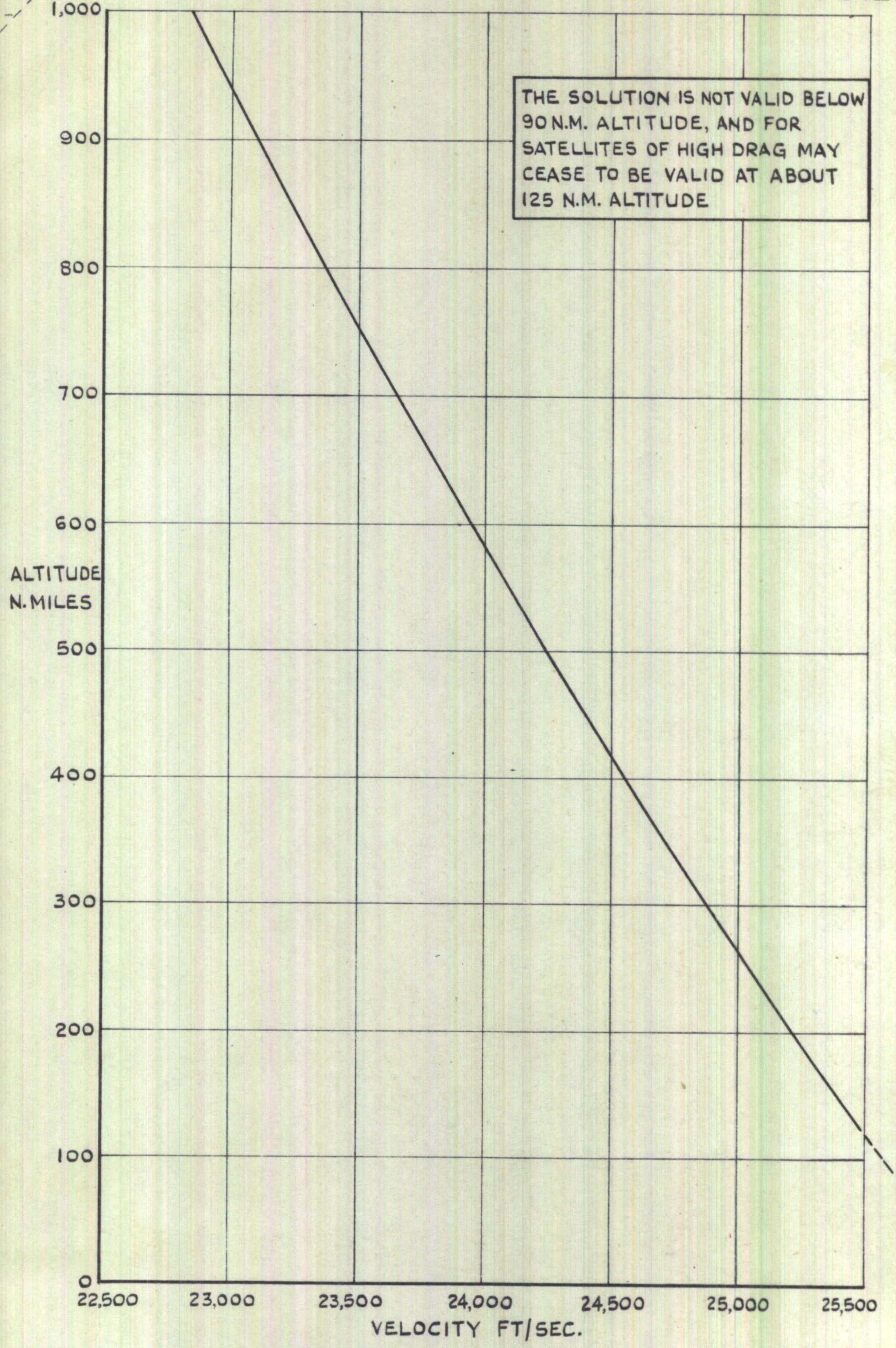
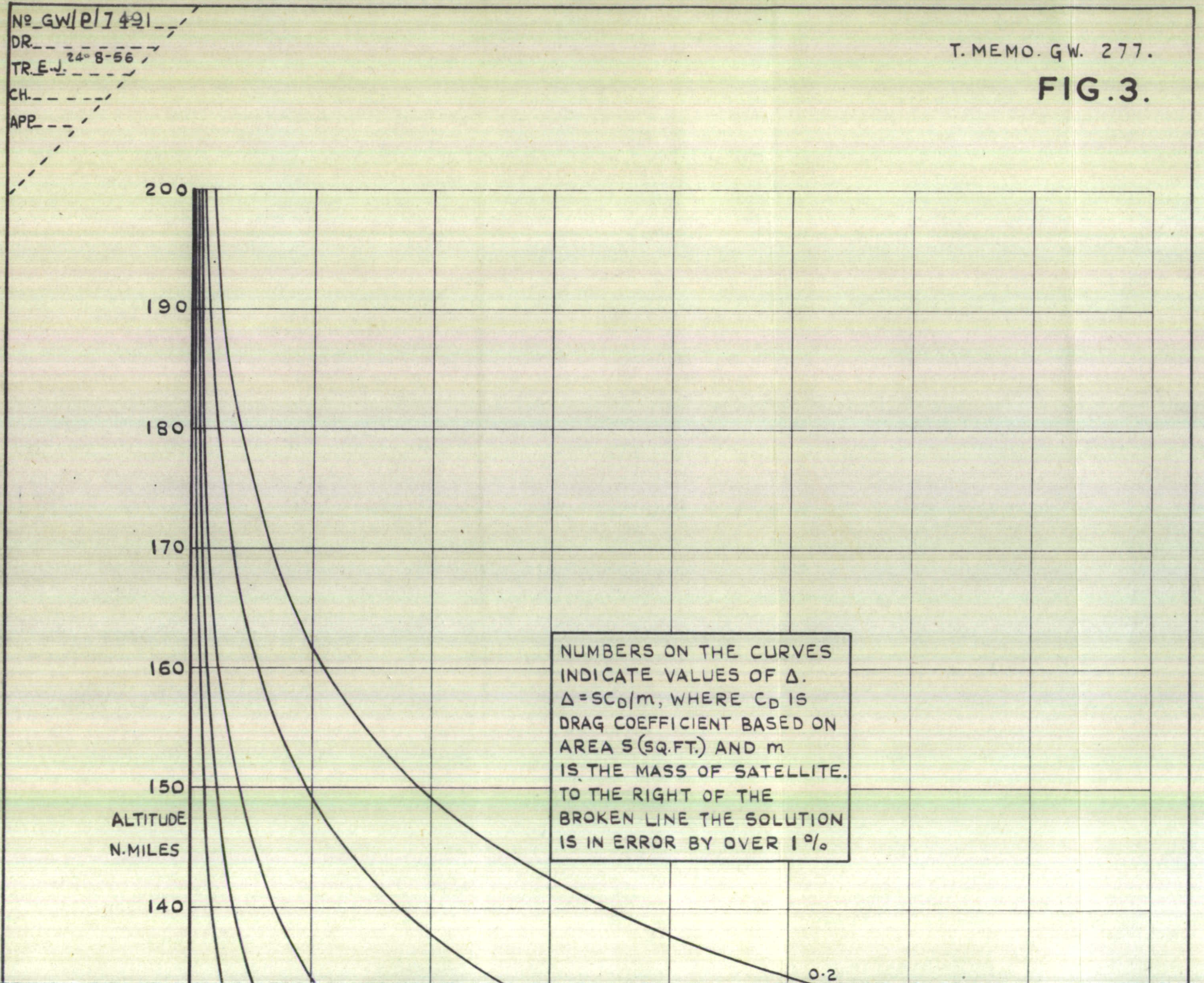


FIG. 2. VELOCITY OF A SATELLITE DURING ITS DESCENT.

FIG. 3.



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200
190
180
170
160
150
ALTITUDE
N. MILES
140
0·2
NUMBERS ON THE CURVES
INDICATE VALUES OF Δ.
Δ = SC_D/m, WHERE C_D IS
DRAG COEFFICIENT BASED ON
AREA S (SQ.FT.) AND m
IS THE MASS OF SATELLITE.
TO THE RIGHT OF THE
BROKEN LINE THE SOLUTION
IS IN ERROR BY OVER 1%

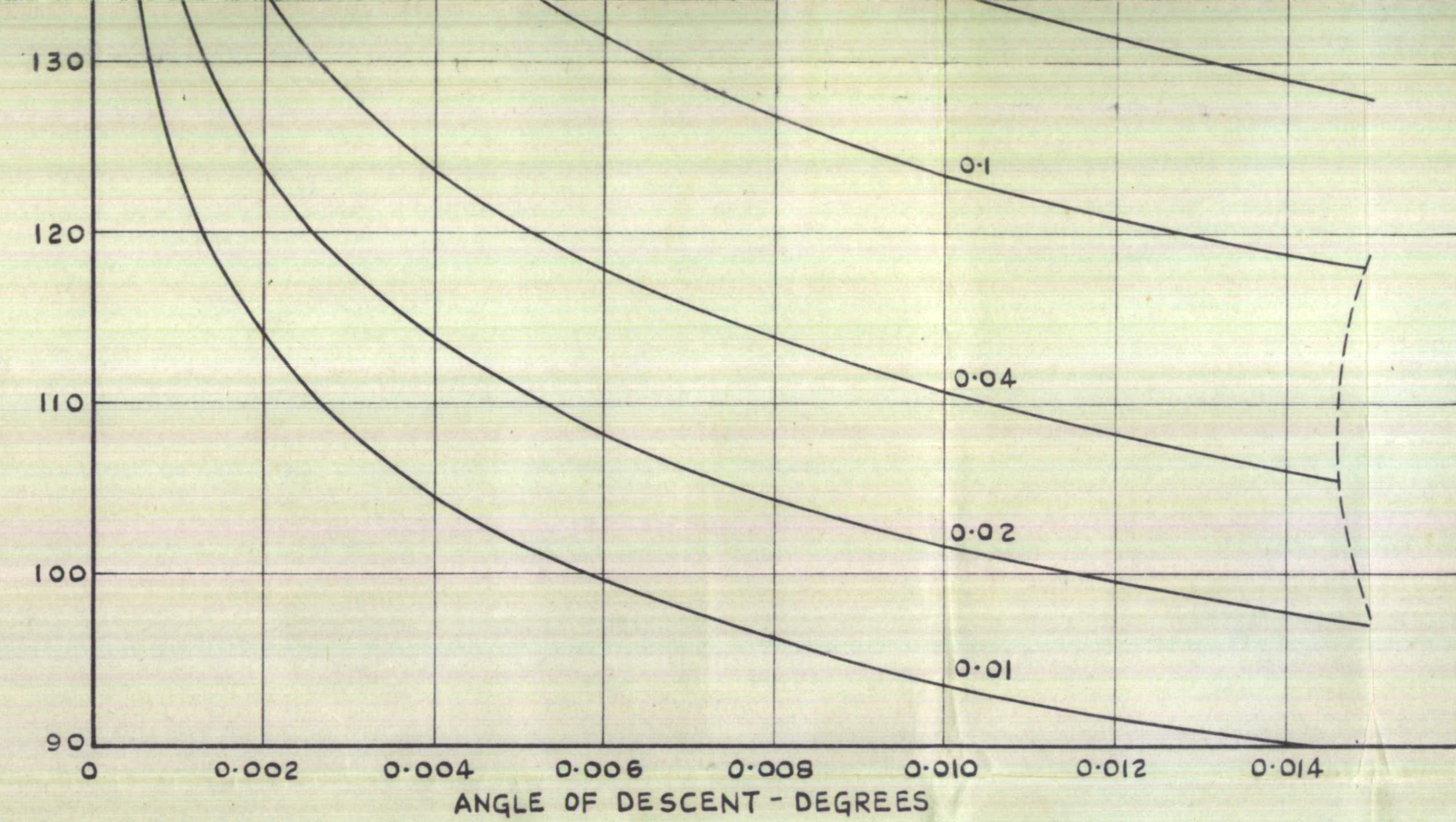

FIG. 3. VARIATION OF ANGLE OF DESCENT WITH ALTITUDE.

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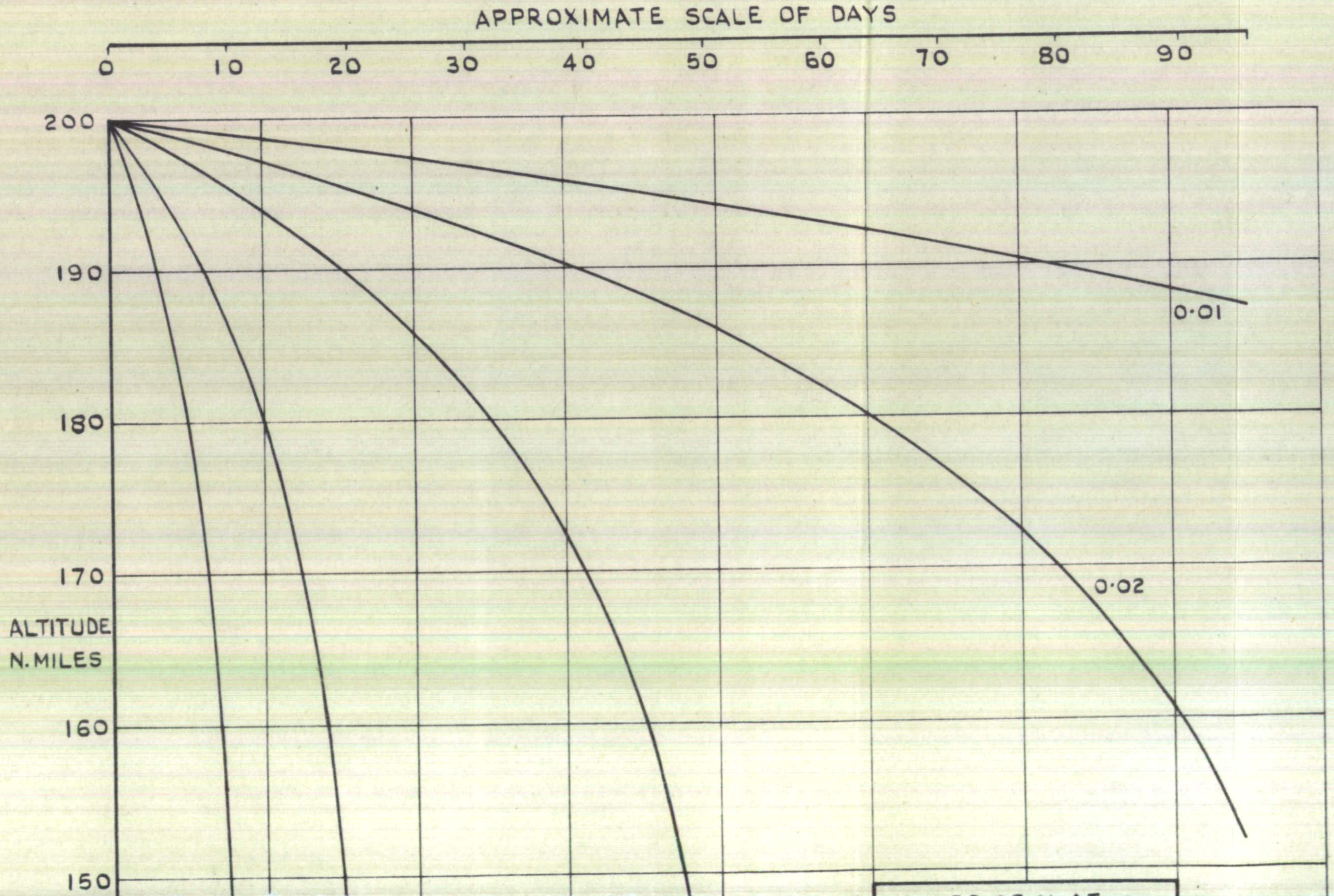
FIG.4.

NUMBERS ON THE CURVES INDICATE VALUES OF Δ.
THE SCALE OF DAYS IS CORRECT FOR Δ = 0·02, AND
MAY BE UP TO 0·8% IN ERROR FOR OTHER VALUES OF Δ



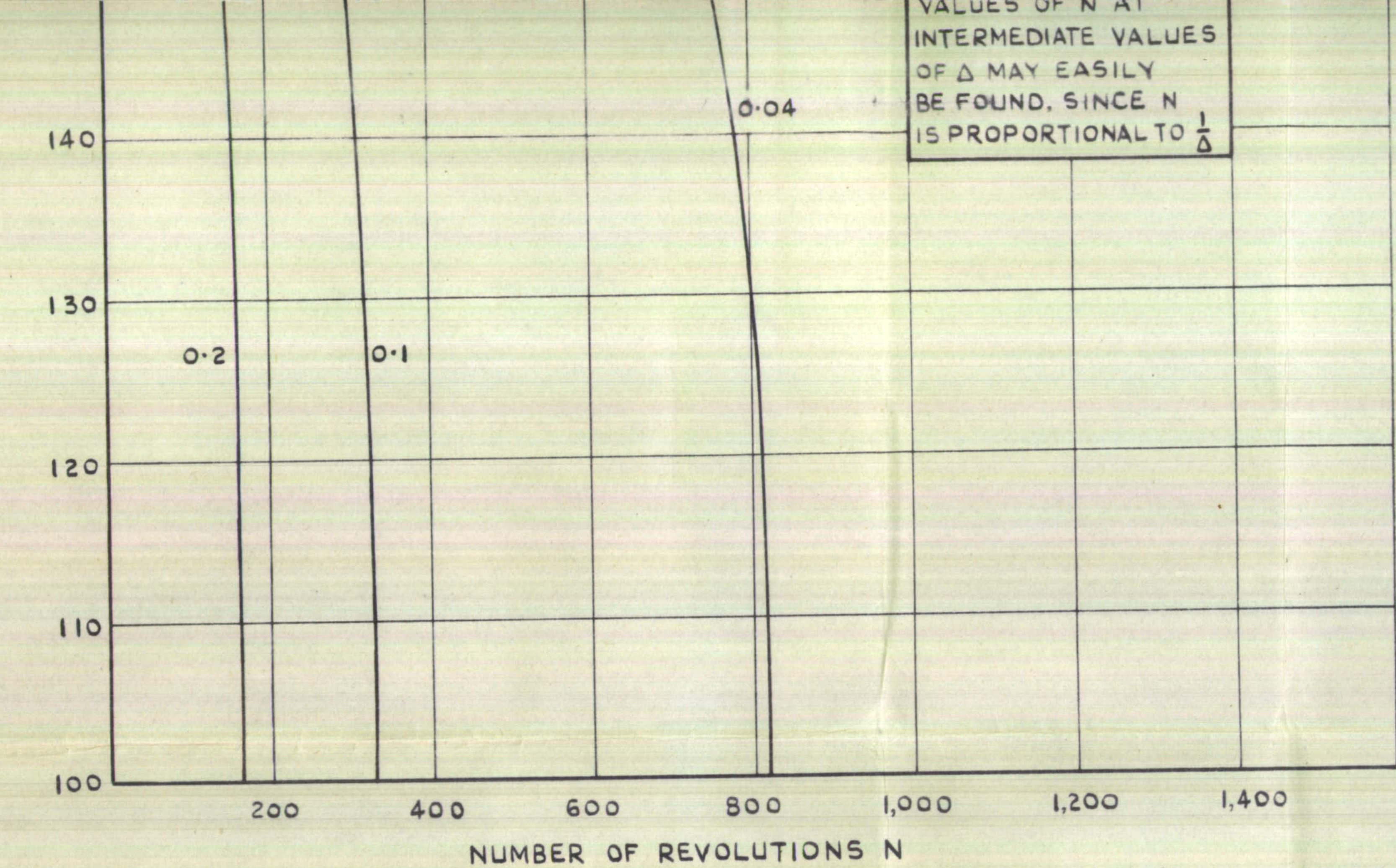


FIG. 4. NUMBER OF REVOLUTIONS MADE IN DESCENDING FROM 200 N. M. ALTITUDE.

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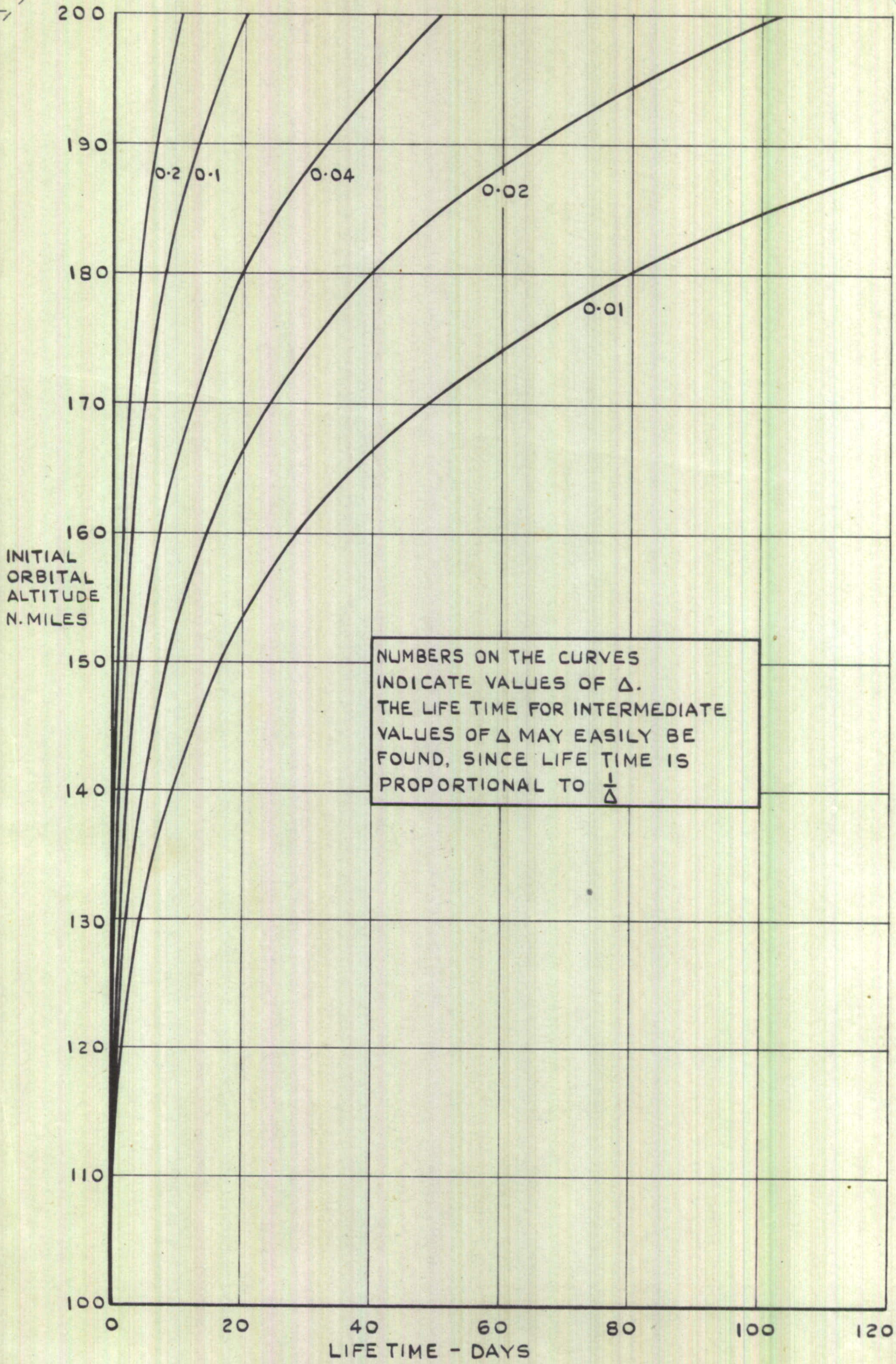

FIG. 5. VARIATION OF LIFETIME WITH INITIAL ORBITAL ALTITUDE.

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Record Summary: AVIA 6/18897
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